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Study of a new semi-continuous model of the Boltzmann equation

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Abstract

For a semi-continuous model of the Boltzmann equation (1) peculiar solutions are obtained and generally the global existence of solutions of the initial value problem is discussed. The global existence is possible even in some cases for partially negative initial number densities, which are not physical problems, but mathematical ones. It can be shown that in some cases the entropy begins to increase, reaches a maximum and decreases again.

Keywords: Boltzmann equation; Semi-continuous model; Global existence

1. Introduction

The purpose of our work is the study of a semi-continuous model of the Boltzmann equation. The name semi-continuous means that the moduli of velocities are discrete, however the directions vary continuously. The following equation is to be studied:

$$\frac{\partial N(t;\theta)}{\partial t} = \frac{1}{2\pi}\int_0^{2\pi}\left\{N(t;\phi)N(t;\phi+\pi) - N(t;\theta)N(t;\theta+\pi)\right\}\mathrm{d}\phi. \tag{1}$$

This equation is obtained by a limiting process from the discrete kinetic theory [1]. The unknown function $N(t;\theta)$, a number density, is periodic in θ, with the period 2π, and depends on the two independent variables, t, time, and θ, an angle. The right-hand side of Eq. (1) represents the collisions terms; the Maxwellian solutions, or equilibrium solutions are those for which the right-hand side is zero.

If the initial data $N(0;\theta)$ have the period $2\pi/n$, then the solutions also possess the same period: $N\{t;\theta+2\pi/n\} = N(t;\theta)$. For any function $F(\cdot\,;\theta)$, we introduce the mean value on the interval $0 < \theta < 2\pi$:

$$\left\langle F(\cdot\,;\theta)\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(\cdot\,;\theta)\,\mathrm{d}\theta. \tag{2}$$

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$\langle N(t;\theta)\rangle$ is by definition independent of θ, but it is also independent of t, as a consequence of Eq. (1); then we will put $\langle N(t;\theta)\rangle = \langle N(0;\theta)\rangle = \langle N\rangle$. One can also derive the following relation:

$$\frac{1}{\pi}\int_0^{\pi} N(t;\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\int_{\pi}^{2\pi} N(t;\theta)\,\mathrm{d}\theta = \langle N\rangle. \tag{3}$$

When the function $N(0;\theta)$ has the period π in θ: $N(0;\theta+\pi) = N(0;\theta)$, then the solution of Eq. (1) has the same period, and the general solution has been already obtained and discussed in previous investigations [4,5]. The solution is given, in a parametric form, by the formulas (6) and (7), where Q_∞ is a constant, and:

$$n_0(\theta) = \frac{N(0;\theta) - \langle N\rangle}{\langle N\rangle}, \qquad a = \inf n_0(\theta), \qquad b = \sup n_0(\theta).$$

As $\langle n_0(\theta)\rangle = 0$ when $N(0;\theta)$ is a constant and not zero, a is strictly negative and b strictly positive.

Using the following formulas:

$$\psi(z) = \exp\bigl\{2\bigl\langle \mathrm{Log}\bigl[1 + z n_0(\theta)\bigr]\bigr\rangle\bigr\}, \tag{4}$$

$$Q(z) = 2Q_\infty \int_0^z \exp\bigl\{-2\bigl\langle \mathrm{Log}\bigl[1 + z n_0(\theta)\bigr]\bigr\rangle\bigr\}\,\mathrm{d}z. \tag{5}$$

$$2\langle N\rangle t(z) = -\,\mathrm{Log}\left\{1 - 2\int_0^z \exp\bigl\{-2\,\mathrm{Log}\bigl[1 + n_0(\theta)\bigr]\bigr\}\,\mathrm{d}z\right\}, \tag{6}$$

$$\frac{N^*(z;\theta)}{\langle N\rangle} = 1 + \psi(z)\left\{1 - \frac{Q(z)}{Q\infty}\right\}\left\{\frac{n_0(\theta)}{1 + z n_0(\theta)} - \left\langle \frac{n_0(\theta)}{1 + z n_0(\theta)}\right\rangle\right\}. \tag{7}$$

We have proved [7,8] the conjecture on “eternal” positive solutions: the only “eternal” positive solutions are constants, which, for the case of solutions with period π, are the Maxwellian solutions.

The purpose of the present paper is to study the solutions of Eq. (1), when the initial data have the period 2π, but not the period π. We will consider models which in the physical plane are symmetric versus the abscissa, $\theta = 0$; the initial data always satisfies the condition $N(0;2\pi - \theta) = N(0;\theta)$, and as a consequence we get: $N(t;2\pi - \theta) = N(t;\theta)$. Therefore Eq. (1) can be replaced by the following equation:

$$\frac{\partial N(t;\theta)}{\partial t} = \frac{1}{\pi}\int_0^{\pi} \bigl\{N(t;\phi)N(t;\phi+\pi) - N(t;\theta)N(t;\theta+\pi)\bigr\}\,\mathrm{d}\phi. \tag{8}$$

2. Analytical study of solutions

2.1. The Boltzmann *H*-function

We introduce the function $H(t) = H\{N(t;\theta)\}$ by the following relation:

$$H(t) = \frac{1}{2\pi}\int_0^{\pi} \bigl\{N(t;\theta)\,\mathrm{Log}\bigl|N(t;\theta)\bigr| + N(t;\theta+\pi)\,\mathrm{Log}\bigl|N(t;\theta+\pi)\bigr|\bigr\}\,\mathrm{d}\theta$$

which is a functional of the solution $N(t;\theta)$ of Eq. (1). For positive values of $N(t;\theta)$, the function $H(t)$, called the Boltzmann H-function, has a lower bound:

$$H\bigl\{N(t;\theta)\bigr\} = \frac{1}{2\pi}\int_0^{2\pi} N(t;\theta)\,\mathrm{Log}\bigl|N(t;\theta)\bigr|\,\mathrm{d}\theta \geqslant -\frac{1}{\mathrm{e}}. \tag{9}$$

Putting $N(t;\theta)N(t;\theta+\pi) = X(t;\theta)$, and using the relation (8), we obtain the following relations (10) and (11):

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \bigl\{2+\operatorname{Log}\bigl|X(t;\theta)\bigr|\bigr\}\bigl\{X(t;\phi)-X(t;\theta)\bigr\}\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{10}$$

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \bigl\{2+\operatorname{Log}\bigl|X(t;\phi)\bigr|\bigr\}\bigl\{X(t;\theta)-X(t;\phi)\bigr\}\,\mathrm{d}\phi\,\mathrm{d}\theta. \tag{11}$$

By subtracting (11) from (10) we get the following relation:

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \left\{\operatorname{Log}\left|\frac{X(t;\theta)}{X(t;\phi}\right|\right\}\bigl\{X(t;\phi)-X(t;\theta)\bigr\}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{12}$$

The physical case corresponds to positive values of the functions $N(t;\theta)$, because they represent number densities. But Eq. (1) can also have nonphysical solutions for which $N(t;\theta)$ could be negative; this is the case, e.g., if $N(0;\theta) = \langle N\rangle\{1+r\cos(2\theta)\}$, when $1<|r|$, for which we know [4,5] that the solution exists globally towards the future, if furthermore $|r|<8/3$.

When the functions $N(t;\theta)$ are positive, also the functions $X(t;\theta)$ and $X(t;\phi)$ also positive, and the derivative $\mathrm{d}H(t)/\mathrm{d}t$ is negative or null, because the product $(1-y)\operatorname{Log}|y|$, for $y>0$, is negative (or null, if $y=0$). The function $H(t)$ is then decreasing, and as it is greater than $-1/\mathrm{e}$, the final state, $H(+\infty)=H_*$, is an equilibrium. For this state $X(+\infty;\phi) = X(+\infty;\theta)$; and then $(\partial N/\partial t)(+\infty;\theta)=0$. A similar result is valid for $t=-\infty$, when those solutions exist globally towards the past. It is also possible that, for some values of the time, the function $H(t)$ has increasing character; of course this corresponds to times for which the function $N(t;\theta)$ is partially negative.

2.2. Maxwellian solutions

The two functions $N(t;\theta)$ and $N(t;\theta+\pi)$ have the same derivative with respect to the time, and therefore $N(t;\theta+\pi) - N(t;\theta) = h(\theta)$, the function $h(\theta)$ being determined by the initial conditions. Furthermore, for the positive values of $N(t;\theta)$, the product $N(+\infty;\theta)N(+\infty;\theta+\pi) = k^2$ is independent of θ. Then

$$N(+\infty;\theta) = -h(\theta) + \sqrt{k^2+h^2(\theta)}. \tag{13}$$

These result is an analytical result for the solutions with period π. For the solutions with period 2π, it is a numerical one. In fact, all the following results concerning the solutions with period 2π, are a consequence of the constant value of the product $N(+\infty;\theta)N(+\infty;\theta+\pi)$. In all examples given, this product is constant, and we have obtained no example for which this product is not constant.

When the solution $N(t;\theta)$ has the period π, the equilibrium solutions, Maxwellian solution, is constant. When the solution has the period 2π, but not the period π, the solution, Maxwellian solution, is not constant and given by formula (13).

2.3. Peculiar solutions

The function $h(\theta)$ has the property $h(\theta+\pi) = -h(\theta)$, and the function $h^2(\theta)$ has the period π. The function $M(t;\theta) = N(t;\theta)+h(\theta)$ has also the period π, and satisfies the equation:

$$\frac{\partial M(t;\theta)}{\partial t} = \frac{1}{2\pi}\int_0^{2\pi}\bigl\{M^2(t;\phi)-M^2(t;\theta)\bigr\}\,\mathrm{d}\theta + b(\theta), \tag{14}$$

with

$$b(\theta) = h^2(\theta) - \frac{1}{\pi}\int_0^\pi h^2(\phi)\,\mathrm{d}\phi. \tag{15}$$

The mean value $\langle b(\theta)\rangle$ is equal to zero. When the function $b(\theta)$ is also equal to zero, the Eq. (14) is the same as Eq. (1), for the unknown functions with period π; the general solution is given by formulas (6) and (7). Using these remarks, it is easy to obtain peculiar solutions for Eq. (1), which have the period 2π, but not the period π. To obtain such a solution it is necessary, and of course sufficient, that $h^2(\theta)$ should be constant, which corresponds to models, called "nonsymmetrical toothed wheels." Figs. 1 and 2, represent one of those models for the times $t=0$ and $t=+\infty$; the model represented corresponds to the following solution:

$$N(t;\theta)=\langle N\rangle\frac{1-3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}-\langle N\rangle\frac{2r}{1-r},\quad \text{for } 0^\circ<\theta<90^\circ,$$

$$N(t;\theta)=\langle N\rangle\frac{1-3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}+\langle N\rangle\frac{2r}{1-r},\quad \text{for } 90^\circ<\theta<120^\circ,$$

$$N(t;\theta)=\langle N\rangle\frac{1+3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}+\langle N\rangle\frac{2r}{1-r},\quad \text{for } 120^\circ<\theta<180^\circ,$$

$$N(t;\theta)=\langle N\rangle\frac{1-3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}+\langle N\rangle\frac{2r}{1-r},\quad \text{for } 180^\circ<\theta<270^\circ,$$

$$N(t;\theta)=\langle N\rangle\frac{1-3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}-\langle N\rangle\frac{2r}{1-r},\quad \text{for } 270^\circ<\theta<300^\circ,$$

$$N(t;\theta)=\langle N\rangle\frac{1+3r\mathrm{e}^{-2\langle N\rangle t}}{1-r\mathrm{e}^{-2\langle N\rangle t}}-\langle N\rangle\frac{2r}{1-r},\quad \text{for } 300^\circ<\theta<360^\circ.$$

As $\langle N\rangle$ is positive, the limits $N(+\infty;\theta)$ and $N(-\infty;\theta)$ are:

$$N(+\infty;\theta)\;=\;\langle N\rangle\frac{1+3r}{1-r},\quad \text{for } 0^\circ<\theta<90^\circ, \tag{16}$$

$$N(+\infty;\theta)\;=\;\langle N\rangle\frac{1+r}{1-r},\quad \text{for } 90^\circ<\theta<270^\circ, \tag{17}$$

$$N(+\infty;\theta)\;=\;\langle N\rangle\frac{1-3r}{1-r},\quad \text{for } 270^\circ<\theta<300^\circ, \tag{18}$$

and

$$N(+\infty;\theta)N(+\infty;\theta+\pi)=\langle N\rangle^2\frac{(1+r)(1-3r)}{(1-r)^2},\qquad N(-\infty;\theta)N(-\infty;\theta+\pi)=\langle N\rangle^2\frac{(3-r)(3-5r)}{(1-r)^2}.$$

2.4. Nonsymmetrical daisies

If we choose the initial data according to the formula:

$$N_n(0;\theta)=\langle N\rangle\bigl\{1+r\cos(n\theta)\bigr\}, \tag{19}$$

the solutions are given by nonsymmetrical daisies. When n is even, $n=2m$, $m\in N$, the solution has the period π, and has been computed explicitely [4,5] in the form:

$$2\langle N\rangle t(z)=-\mathrm{Log}\left\{1-\frac{4}{r}\left[\frac{z}{1+\sqrt{1-z^2}}-\frac{1}{3}\left(\frac{z}{1+\sqrt{1-z^2}}\right)^3\right]\right\}, \tag{20}$$

$$\frac{N_n^*(z;\theta)}{\langle N\rangle}=1+\frac{E(z)}{4}\left\{\frac{r\cos(2m\theta)}{1+Z\cos(2m\theta)}+\frac{r}{1-z^2+\sqrt{1-z^2}}\right\}, \tag{21}$$

with

$$E(z)=\bigl(1+\sqrt{1-z^2}\bigr)^2\exp\bigl\{-2\langle N\rangle t(z)\bigr\}. \tag{22}$$

For $r<8/3$, the solution is global in time towards the future; for $r>8/3$ the solution blows up at the time t_e^+, which is independent of m:

$$2t_e^+=-\mathrm{Log}\left(1-\frac{8}{3r}\right). \tag{23}$$

When the time is infinite, the limit $N(+\infty;\theta)=\langle N\rangle$ is constant, independent of θ, and of m.

From the formulas (20) and (21) one gets:

$$N_{2m}(t;\theta)=N_2(t;m\theta). \tag{24}$$

In Fig. 5 as an initial distribution a daisy with 6 petals is represented. For $t=+\infty$ the daisy is reduced to a circle.

When n is odd, $n=2m+1$, $m\in N$, we arrive at a similar relation:

$$N_{2m+1}(t;\theta)=N_1\bigl\{t;(2m+1)\theta\bigr\}. \tag{25}$$

To obtain this result, it is sufficient to replace in Eq. (1) θ by $(2m+1)\theta$, $\theta+\pi$ by $(2m+1)(\theta+\pi)$, ϕ by $(2m+1)\phi$, and $\phi+\pi$ by $(2m+1)(\phi+\pi)$. To study the nonsymmetrical daisies, it is sufficient to compute the solution $N_1(t;\theta)$. Unfortunately we have not been able to find any relation between $N_1(t;\theta)$ and $N_2(t;\theta)$.

3. Numerical study

3.1. Global existence

For the symmetrical case, that means for the solutions with period π of Eq. (1), when the initial data are given, if the solution $N(t;\theta)$ exists globally in time towards the future and towards the past, there are "eternal" solutions given analytically by the formulas (4)–(7). We recall the results, which depend upon the minimum a and of the maximum b of the ratio $\{N(0;\theta) - \langle N\rangle\}/\langle N\rangle$, and of the value z_∞, defined by relation:

$$1 - 2\int_0^{z_\infty} \exp\bigl\{-2\bigl\langle \mathrm{Log}\bigl[1 + \xi n_0(\theta)\bigr]\bigr\rangle\bigr\}\,\mathrm{d}\xi = 0. \tag{26}$$

Theorem 1. *Let $N(0;\theta+\pi) = N(0;\theta)$. The initial value problem for Eq.* (1) *possesses a solution which is global in time extending to the future* $(t > 0)$ *if and only if the following inequality is satisfied*:

$$z_\infty < -\frac{1}{a}.$$

Theorem 2. *Let $N(0;\theta+\pi) = N(0;\theta)$. The initial value problem for Eq.* (1) *possesses a solution which is global in time extending to the past* $(t < 0)$ *if and only if*

$$N(0;\theta) = b \tag{26}$$

is valid on intervals the sum of which is greater or equal π.

For the nonsymmetrical case, as we did not succeed to obtain the analytical solution to Eq. (1), we have performed numerical integrations [6] using the Gauss method to compute the integral term (which represent the collisions),

$$\int_{-1}^{1} f(t;\theta)\,\mathrm{d}\theta \quad \text{is approximated by} \quad \sum_j \frac{2(1-\theta_j)^2}{n^2 P_{n-1}(\theta_j)}, \tag{27}$$

with

$$P_n(x) = \frac{1}{2\times 4\times\cdots\times(2n)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl(x^2-1\bigr)^n. \tag{28}$$

The polynomials $P_n(x)$ are the Legendre polynomials, and j varies from 1 to 100. The initial data $N(0;\theta)$ are defined by their Fourier series, and the program computes the solution $N(t;\theta)$. Essentially, the cases of nonsymmetrical daisies were studied, for which the following equation holds:

$$N_r(0;\theta) = \langle N\rangle\bigl\{1 + r\cos(n\theta)\bigr\}. \tag{29}$$

The maximum value of r for which the solutions $N_r(t;\theta)$ exist globally towards the future only depends on the parity of n; it is $r = 8/3$ for even values of n. These results are a consequence of relations (20)–(22). When the solutions do not exist globally towards the future, the times t_e^+ of explosion, for n even, is given by relation (23).

If n is an odd number, for $r < 1.569\ldots$ the solution is global in time. For $r > 1.569\ldots$, the times t_e^+ which are calculated by numerical integrations are given in Tables 1 and 2 and are compared with values corresponding to the case when n is even.

The limit $N(+\infty;\theta)$ is $\langle N\rangle$, when n is even. When n is odd the limit is given by formula (13) where $h(\theta) = -\langle N\rangle r\cos(n\theta)$; $k^2 = \langle N\rangle^2 f^2(r)$, is independent of n, and:

$$N(+\infty;\theta) = \langle N\rangle\Bigl\{r\cos(n\theta) + \sqrt{f^2(r) + r^2\cos^2(n\theta)}\Bigr\}. \tag{30}$$

When n is odd, for small values of r one gets

$$f^2(r) = 1 - \frac{r^2}{2} + \mathrm{O}\bigl(r^4\bigr), \tag{30}$$

and:

$$N(+\infty;\theta) = \langle N\rangle\Bigl\{1 + r\cos(n\theta) + \frac{r^2}{4}\cos(2n\theta) + \cdots\Bigr\}. \tag{30}$$

Table 1
Times of explosion when $N(0;\theta) = \langle N\rangle\{1 + r\cos(n\theta)\}$

r	t_e^+, if n is even	t_e^+, if n is odd
9	0.176	0.322
7	0.240	0.431
5	0.382	0.658
3	1.106	1.379
2.6	global existence	1.787
2.2		2.586
1.8		5.186
1.4		global existence

Table 2
Values of $f^2(r)$, when $N(0;\theta) = \langle N\rangle\{1 + r\cos(n\theta)\}$

r	f^2, if n is even	f^2, if n is odd
0.0	1.000	1.000
0.4	1.000	0.921
0.8	1.000	0.693
1.2	1.000	0.348
1.569	1.000	0.000

3.2. Entropy

The entropy is defined by relation (9), and its derivative is given by relation (12). When the function $N(t;\theta)$ is positive the derivative (12) is negative. If we consider not the physical problem, but only a mathematical problem, the functions $N(t;\theta)$ can be partially negative, for example, if one chooses the initial data $N(0;\theta)$ partially negative. It is the case, for example, for the daisies $N(0;\theta) = \langle N\rangle\{1 + r\cos(n\theta)\}$, with $|r| < 1$, nevertheless the solution exists globally towards the future. For these examples it is possible that, near the initial time, the entropy could be increasing. Table 3 give the values of entropy, near the initial time, for the daisies with $r = 1.5$.

The values of $H(t)$ are the same for all even values of n; they are also the same, but different for all odd values of n. When n is even the entropy $H(t)$ is always decreasing. When n is odd the entropy $H(t)$ is always decreasing for $r < 1.414$; for $r > 1.414$ the entropy begins to increase, has a maximum, and then decreases; those cases correspond, of course, to partially negative initial data $N(0;\theta)$. For $r = 1.5$, and n odd, the maximum, reached for $t = 7.60$, is $H(7.60) = 4.62744\ldots$. When the time t becomes infinite $H(t)$ tends to 0, if n is even; when n is odd, the limit value of $H(t)$ is a function of r:

r	$H(\infty)$
0.0	0.000
0.5	0.398
1.0	1.689
1.5	4.465

Table 3
Values of $H(t)$ for $N(0;\theta) = \langle N\rangle\{1 + r\cos(n\theta)\}$; $r = 1.5$

t	$H(t)$, if n is even	$H(t)$, if n is odd
0	4.463	4.476
2	2.358	4.548
4	0.990	4.598
6	0.453	4.622
8	0.212	4.627
10	0.099	4.619
⋮	⋮	⋮
∞	0.000	1.465

All these results are proven by formulas (20) and (21), when n is even; they have been obtained numerically when n is odd.

3.3. Explanation of the figures

Figs. 1 and 2 represent a solution called toothed wheel, given in Section 2.3, at initial time (Fig. 1) and when the time is infinite (Fig. 2).

Fig. 3 represents the solutions $N(t;\theta)$ as function of the time t, when $N(0;\theta) = \langle N\rangle\{1 + 8\cos(\theta)\}$ for different values of θ.

Fig. 4 represents the solutions $N(t;\theta)$ as functions of the time t, when $N(0;\theta) = \langle N\rangle\{1 + r\cos(\theta)\}$ for different values of r, θ being fixed.

Fig. 5 represents the solution $N(t;\theta)$, when $N(0;\theta) = \langle N\rangle\{1 + 0.2\cos(6\theta)\}$, daisy with 6 petals, at times $t = 0$ and $t = +\infty$.

Fig. 6 represents the solution $N(t;\theta)$, when $N(0;\theta) = \langle N\rangle\{1 + \cos(3\theta)\}$, daisy with 3 petals, at times $t = 0$ and $t = +\infty$.

Fig. 7 represents the solution $N(t;\theta)$, when $N(0;\theta) = \langle N\rangle\{1 + 0.2\cos(5\theta)\}$, daisy with 5 petals, at times $t = 0$ and $t = +\infty$.

Fig. 8 represents the solution $N(t;\theta)$, when $N(0;\theta) = \langle N\rangle\{1 + 0.9\cos(5\theta)\}$, daisy with 5 petals, at times $t = 0$ and $t = +\infty$.

4. Conclusion

The results we have obtained concern the nonsymmetrical solutions of Eq. (1), which is a semi-continuous model of the Boltzmann equation. First, we have built a family of peculiar solutions: the models of toothed wheels. Secondly, we obtained some analytical properties for the nonsymmetrical solutions: the behavior when the time increases indefinitely, the form of the Maxwellian solutions, which are not constant, contrary to the case of symmetrical solutions. Thirdly we have studied, both analytically and numerically the models of daisies: $N(0;t) = \langle N\rangle\{1 + r\cos(n\theta\}$, where n is odd; the times of explosions are

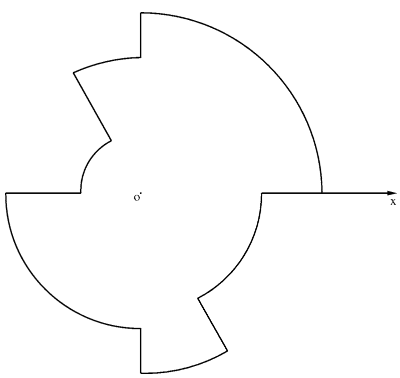

Fig. 1. Toothed wheel, $t = 0$.

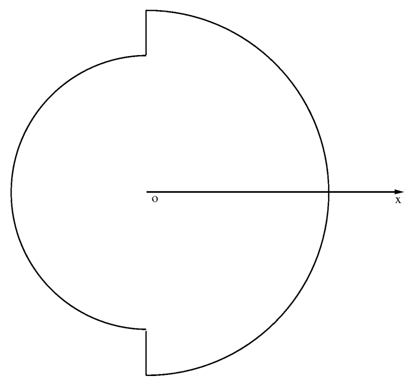

Fig. 2. Toothed wheel, $t = +\infty$.

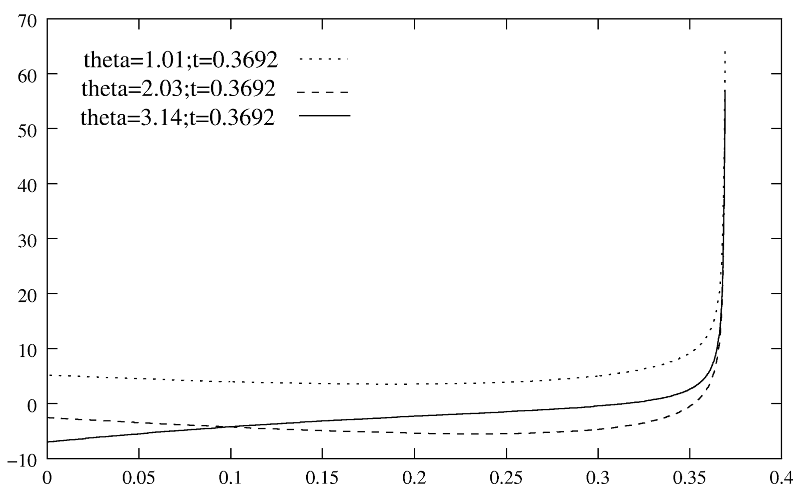

Fig. 3. $N(0;\theta)$ for some values of θ, $N(0;\theta) = \langle N\rangle\{1 + 8\cos(\theta)\}$.

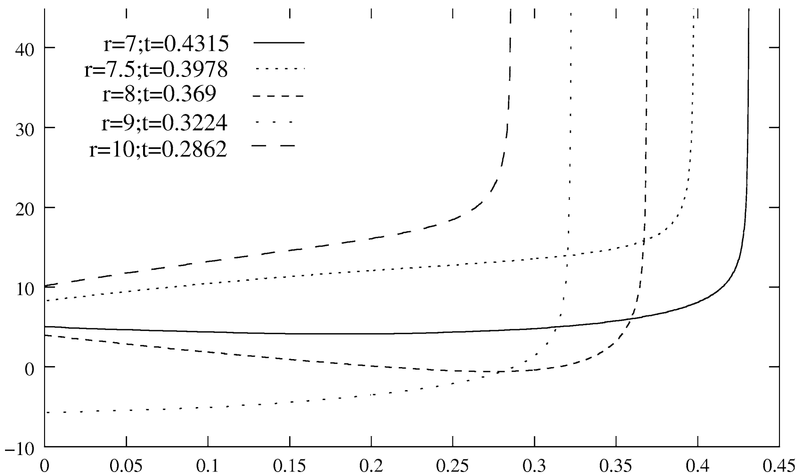

Fig. 4. $N[0;\pi/4]$ for some values of r, $N(0;\theta) = \langle N\rangle\{1 + r\cos(\theta)\}$.

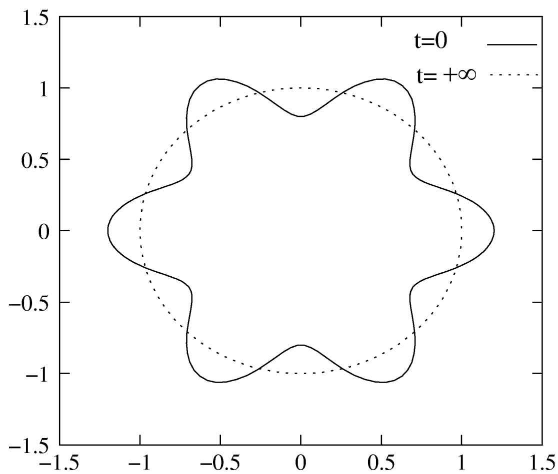

Fig. 5. Daisy with 6 petals, $N(0;\theta) = \langle N\rangle\{1 + 0.2\cos(6\theta)\}$.

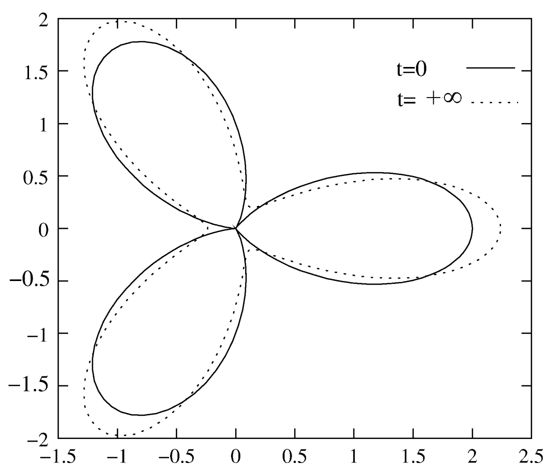

Fig. 6. Daisy with 3 petals, $N(0;\theta) = \langle N\rangle\{1 + \cos(3\theta)\}$.

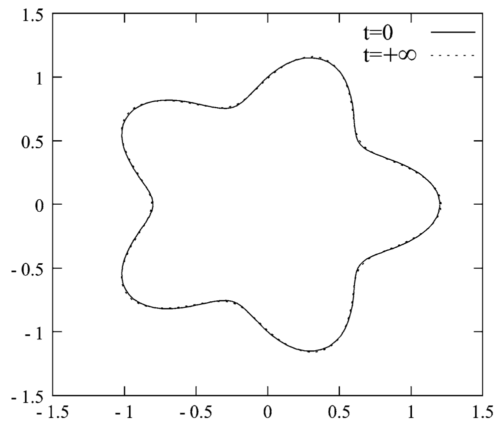

Fig. 7. Daisy with 5 petals, $N(0;\theta) = \langle N\rangle\{1 + 0.2\cos(5\theta)\}$.

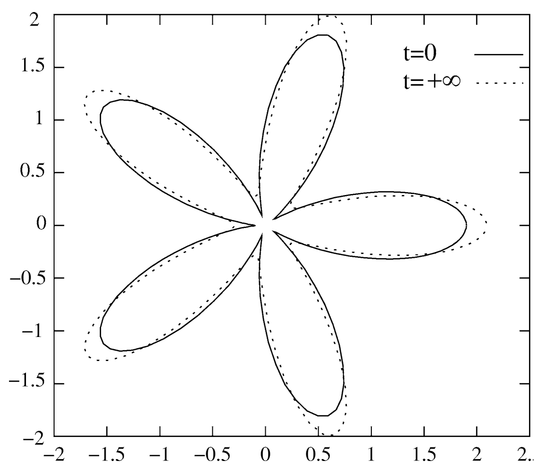

Fig. 8. Daisy with 5 petals, $N(0;\theta) = \langle N\rangle\{1 + 0.9\cos(5\theta)\}$.

independent of the number n of petals, similarly to the symmetrical cases, when n is even, but in both the cases the values, which depend only on r are not the same. In the first case the explosions appear for $r = 1.569\ldots$, in the second case for $r = 8/3$. At the end of our investigation the entropy was computed, which is—as it is well-known—a decreasing function of the time when the functions $N(t;\theta)$ are positive. This corresponds to the physical problem because those functions represent densities; but in the mathematical problem the functions $N(t;\theta)$ can be partially negative, in particular at the initial time if $|r| > 1$. For some nonsymmetrical cases the entropy can begin to increase, reach a maximum, and the decreases. For both the cases—symmetrical and nonsymmetrical—it is possible to have global existence of the solutions for initial data which are partially negative; this result is a new generalisation of former results, which were obtained for the Carleman model [2], and for the Broadwell model [3].

The mathematical proofs for all these results—obtained numerically in the present investigation—are of course still open problems.

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